Orbital Glider Range Maximization

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Introduction

The problem of finding the angle-of-attack program $\alpha(t)$ for a hypersonic glider to achieve maximum range is investigated. The purpose of this Note is to compare the solution of the problem by the method of variation of extremals (or second variations) with the steepest-ascent method used by Bryson and Denham.[1] The use of rectangular velocity components in preference to the polar components of earlier work has yielded an improved determination of the control program. The resulting optimum trajectory has about four times the range found earlier and many more "skips." The extensive porpoising to get into a region of low drag appears to reflect the near-orbital character of the trajectory.

Problem Statement

The nomenclature for the problem is shown in Fig. 1. Following Bryson and Denham[1] the initial conditions used were $V = 25{,}920$ ft sec^{-1}, $\gamma = 0.18°$, and $h = 300{,}000$ ft. The wing loading used was $w = mg_0/S = 27.3$ lb ft^{-2}, where m is the mass of the vehicle, g_0 is the acceleration of gravity at the earth's surface, and S is the wing plan-form area. The Newtonian drag and lift functions used were

$$D = \tfrac{1}{2}\rho(h)V^2SC_D(\alpha),\ L = \tfrac{1}{2}\rho(h)V^2SC_L(\alpha)$$

$$C_D = 0.042 + 1.46|\sin^3\alpha|,\ C_L = 1.82\ \sin\alpha|\ \sin\alpha|\ \cos\alpha$$

where $\rho(h)$ is the density of the atmosphere. The 1959 ARDC model atmosphere values of $\rho(h)$ and the consistent values, $g_0 = 32.174049$ ft sec^{-2} and earth radius $R = 3440$ nautical miles, were obtained from the work of Minzner, Champion, and Pond.[2]

Solution by Extremal Variation

The long flight path involved in this problem makes it desirable to check the steepest-ascent solution by some competing calculus-of-variations method of optimization. The variation of extremals solution given here uses the rectangular velocity components u,v of Fig. 1. These components were found to be preferable to the polar V,γ components in the numerical solution. The differential equations of motion of the glider are

$$\dot{u} = -[\rho(u^2 + v^2)^{1/2}(C_Du + C_Lv)/2w] - uv(R + h)^{-1} \quad (1)$$

$$\dot{v} = -[\rho(u^2 + v^2)^{1/2}(C_Dv - C_Lu)/2w] + u^2(R + h)^{-1} - g \quad (2)$$

$$\dot{h} = v \quad (3)$$

$$\dot{x} = Ru/(R + h) \quad (4)$$

where $g = g_0R^2/(R + h)^2$. The problem is to determine the angle-of-attack program $\alpha(t)$ which will maximize the range $x(T)$ at the terminal time T subject to terminal constraints to be specified later. Since the function to be maximized is a state variable there is no cost function and the Hamiltonian[3] of the system is

$$H = \lambda_u\dot{u} + \lambda_v\dot{v} + \lambda_h\dot{h} + \lambda_x\dot{x} \quad (5)$$

where λ_u, λ_v, λ_h, λ_x are the costate variables corresponding to the state variables u,v,h,x, respectively. Partial differentiation of the Hamiltonian with respect to the state variables yields the adjoint differential equations for the costate variables

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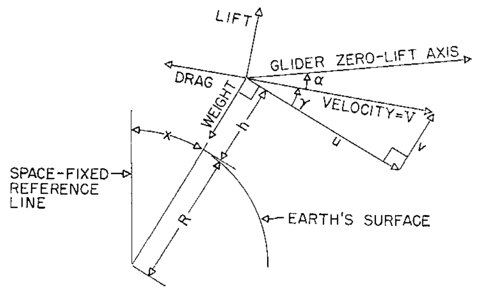

Fig. 1 Hypersonic orbital glider nomenclature.

$$\dot{\lambda}_u = -\partial H/\partial u = \left\{\frac{\rho}{2w}(u^2 + v^2)^{-1/2}[C_D(2u^2 + v^2) + C_Luv] + \frac{v}{R + h}\right\}\lambda_u + \left\{\frac{\rho}{2w}(u^2 + v^2)^{-1/2}[C_Duv - C_L(2u^2 + v^2)] - \frac{2u}{R + h}\right\}\lambda_v - \frac{R}{R + h}\lambda_x \quad (6)$$

$$\dot{\lambda}_v = -\partial H/\partial v = \left\{\frac{\rho}{2w}(u^2 + v^2)^{-1/2}[C_Duv + C_L(2v^2 + u^2)] + \frac{u}{R + h}\right\}\lambda_u + \frac{\rho}{2w}(u^2 + v^2)^{-1/2}[C_D(2v^2 + u^2) - C_Luv]\lambda_v - \lambda_h \quad (7)$$

$$\dot{\lambda}_h = -\partial H/\partial h = \left\{\frac{1}{2w}(u^2 + v^2)^{1/2}(C_D\,u + C_Lv)\frac{d\rho}{dh} - uv(R + h)^{-2}\right\} \times \lambda_u + \left\{\frac{1}{2w}(u^2 + v^2)^{1/2}(C_Dv - C_Lu)\frac{d\rho}{dh} + u^2(R + h)^{-2} + \frac{dg}{dh}\right\}\lambda_v + \frac{Ru\lambda_x}{(R + h)^2} \quad (8)$$

$$\dot{\lambda}_x = -\partial H/\partial x = 0 \quad (9)$$

Let the columns of the matrix

$$\mathbf{\Lambda}_1(t) = \begin{bmatrix} \lambda_{u1} & \lambda_{u2} & \lambda_{u3} & \lambda_{u4} \\ \lambda_{v1} & \lambda_{v2} & \lambda_{v3} & \lambda_{v4} \\ \lambda_{h1} & \lambda_{h2} & \lambda_{h3} & \lambda_{h4} \\ \lambda_{x1} & \lambda_{x2} & \lambda_{x3} & \lambda_{x4} \end{bmatrix} \quad (10)$$

constitute a fundamental system of four linearly independent solutions of Eqs. (6–9) along a glider trajectory defined by Eqs. (1–4), with the initial value $\mathbf{\Lambda}_1(0)$ taken as the unit matrix $\mathbf{I}$. The three columns of the matrix

$$\mathbf{\Lambda}_2(t) = \begin{bmatrix} \lambda_{1u} & \lambda_{2u} & \lambda_{3u} \\ \lambda_{1v} & \lambda_{2v} & \lambda_{3v} \\ \lambda_{1h} & \lambda_{2h} & \lambda_{3h} \\ \lambda_{1x} & \lambda_{2x} & \lambda_{3x} \end{bmatrix} = \begin{bmatrix} \lambda_{u1} & \lambda_{u2} & \lambda_{u3} \\ \lambda_{v1} & \lambda_{v2} & \lambda_{v3} \\ \lambda_{h1} & \lambda_{h2} & \lambda_{h3} \\ \lambda_{x1} & \lambda_{x2} & \lambda_{x3} \end{bmatrix} - \frac{1}{\dot{x}(0)}\begin{bmatrix} \lambda_{u4} \\ \lambda_{v4} \\ \lambda_{h4} \\ \lambda_{x4} \end{bmatrix}[\dot{u}(0)\ \dot{v}(0)\ \dot{h}(0)] \quad (11)$$

satisfy the matrix equation

$$[\dot{u}(0)\ \dot{v}(0)\ \dot{h}(0)\ \dot{x}(0)]\mathbf{\Lambda}_2(0) = \mathbf{0} \quad (12)$$

The three linearly independent columns of $\mathbf{\Lambda}_2(t)$ are needed in the solution to satisfy the requirement (Athans and Falb[3]) that the Hamiltonian H vanish identically along an optimal trajectory, since the right-hand sides of Eqs. (1–4) are not

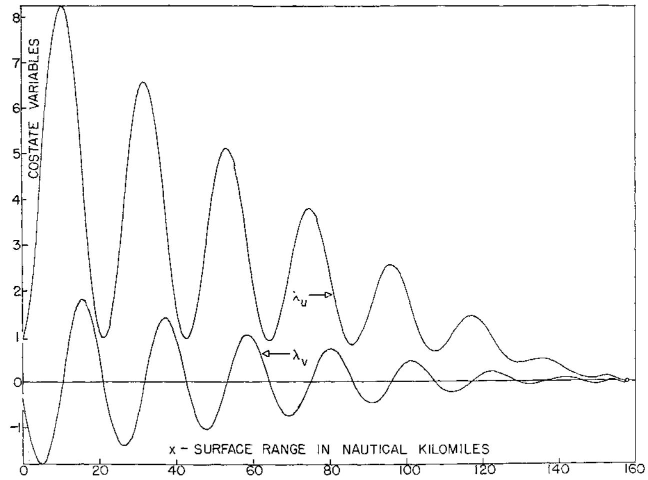

Fig. 2 Costate variables λ_u and λ_v vs range.

explicit functions of time. The costate variables are taken henceforth as the following linear combinations of the columns of Λ_2;

$$\lambda_u = \lambda_{1u} \cos\theta + \lambda_{2u} \sin\theta + \lambda_{3u}e \tag{13}$$

$$\lambda_v = \lambda_{1v} \cos\theta + \lambda_{2v} \sin\theta + \lambda_{3v}e \tag{14}$$

$$\lambda_h = \lambda_{1h} \cos\theta + \lambda_{2h} \sin\theta + \lambda_{3h}e \tag{15}$$

$$\lambda_x = \lambda_{1x} \cos\theta + \lambda_{2x} \sin\theta + \lambda_{3x}e \tag{16}$$

where the weighting factors $\cos\theta$, $\sin\theta$, and e are to be determined from terminal constraints. It is assumed henceforth that the Hamiltonian H of Eq. (5) is constructed from the costate variables of Eqs. (13–16).

The angle-of-attack program $\alpha(t)$ for the desired optimum trajectory is obtained from the maximum principle of Pontryagin[3] which requires that the Hamiltonian H be maximized with respect to α. It is therefore necessary that

$$\partial H/\partial\alpha = 0$$

which yields the following formula for α as a function of θ, e, and t

$$\alpha = \tfrac{1}{2}[\varphi + \arcsin(\tfrac{1}{3}\sin\varphi)] \tag{17}$$

$$\varphi = \arctan[1.82(u\lambda_v - v\lambda_u)/1.46(u\lambda_u + v\lambda_v)] \tag{18}$$

where $-\pi \leq \varphi < \pi$. The functions whose terminal values cannot be chosen freely are $x(T)$ which is to be maximized by a choice of θ and e, and $h(T)$ which we constrain to be 75,000 ft. The state variables with free terminal values are $u(T)$ and $v(T)$. McCue and Good[4] have shown that the freedom of $u(T)$ and $v(T)$ implies the vanishing of the corresponding costate terminal values $\lambda_u(T)$ and $\lambda_v(T)$, since the function to be maximized and the constraint function each involve only one variable of the set of state variables.

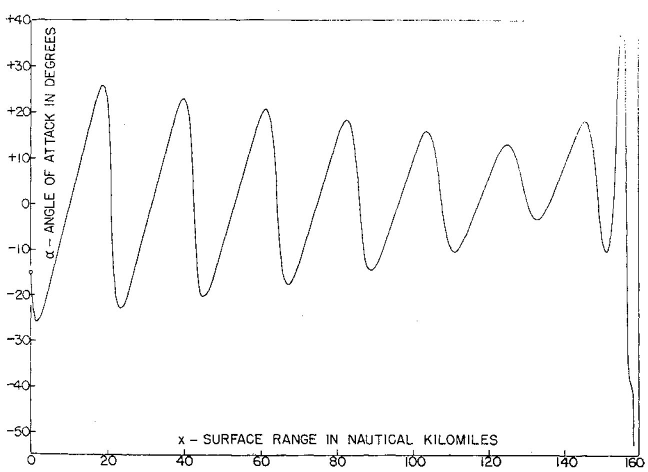

Fig. 3 Optimal angle of attack for maximum range.

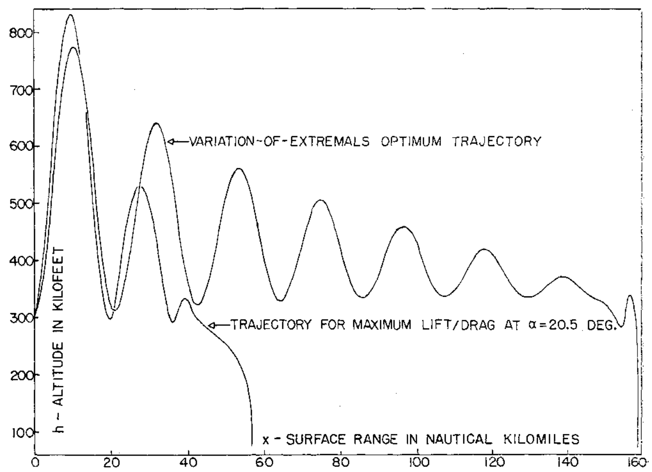

Fig. 4 Altitude vs range for hypersonic orbital glider.

Numerical Solution

The problem is to determine the weighting factors $\cos\theta$, $\sin\theta$, e of Eqs. (13–16) which furnish an admissible solution of the system of Eqs. (1–9 and 17) satisfying the terminal constraints

$$\lambda_u(T) = \lambda_v(T) = 0, \; h(T) = 75{,}000 \tag{19}$$

Initial approximations to θ and e were obtained by integrating the system of equations for $\alpha = 20.5°$, corresponding to the maximum ratio of lift to drag, and then fitting Eqs. (17) and (18) to this α at the arbitrary ranges $x = 10{,}000$ and $x = 15{,}000$ naut miles. It was hoped that these initial approximations could be improved by solving Eqs. (19) by the use of differential corrections $d\theta$ and de obtained from the Newton-Raphson equations

$$\begin{aligned}(\partial\lambda_u/\partial\theta)_T d\theta + (\partial\lambda_u/\partial e)_T de + \dot{\lambda}_u(T)dT &= -\lambda_u(T)\\ (\partial\lambda_v/\partial\theta)_T d\theta + (\partial\lambda_v/\partial e)_T de + \dot{\lambda}_v(T)dT &= -\lambda_v(T) \quad (20)\\ (\partial h/\partial\theta)_T d\theta + (\partial h/\partial e)_T de + \dot{h}(T)dT &= 75{,}000 - h(T)\end{aligned}$$

The partial derivatives in Eqs. (20) were evaluated by integrating the system of differential equations arising from the variation of Eqs. (1–9 and 17). The oscillatory nature of the costate functions λ_u and λ_v, graphed in Fig. 2, shows that the Newton-Raphson method of solution will not be successful unless the approximate values of θ and e are close to the roots of Eqs. (19). This difficulty was surmounted by searching for a ridge in the $x(\theta,e,T)$ surface, and then following the top of the ridge toward the maximum point. As the maximum point is approached, Eqs. (20) begin to be useful in finding the desired admissible solution.

Summary of Results

The optimal solution is given by $\theta = -0.32070$ and $e = 0.0032783$. The corresponding costate functions λ_u and λ_v are graphed in Fig. 2. The angle-of-attack program $\alpha(x)$ is graphed in Fig. 3. The trajectory for the maximum ratio of lift to drag at $\alpha = 20.5°$ and the optimum trajectory are graphed in Fig. 4. The maximum surface range is 158,774 naut miles at $T = 635.35$ min. The numerical integrations were done by the fourth-order Runge-Kutta method with a time step of 5 sec. The error in $x(T)$ is estimated to be less than one mile by a method given by Hildebrand.[5] Single-precision 14-digit and double-precision 28-digit calculations gave the same result. No explanation has been found for the difference between the Bryson and Denham[1] maximum $x(T)$ of 38,500 naut miles and the larger value found here. It is noted that the solution of the problem is very sensitive to various parameters used such as the earth radius and the air density model.

References

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Numerical Solution of Nonlinear Equations for Spinning Shallow Spherical Shells

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Nomenclature

E = Young's modulus
N_{rr}^* = dimensionless radial direct stress resultant, $N_{rr}^* = \gamma Ef/\rho\omega^2R^2x$
R = shell radius of curvature
b = shell peripheral radius
f = nondimensional stress function
g = nondimensional displacement variable
r = radial coordinate
t = shell thickness
x = nondimensional radial coordinate, $x = r/b$
γ = dimensionless shell inertia loading parameter, $(3 + \nu)\rho^2\omega R^2/Et$
Δ = finite difference grid interval
λ^4 = dimensionless shell geometry parameter, $\lambda^4 = 12(1 - \nu^2)(b/R)^2/(t/b)^2$
ν = Poisson's ratio
ρ = mass per unit surface area
σ_{rr}^* = dimensionless radial bending stress resultant, $\sigma_{rr}^* = \{(3 + \nu)t/[12(1 - \nu^2)]^{1/2}\lambda^2\}[dg/dx + g/x]$

Introduction

PREPARATORY to conducting an investigation of the free vibrations of centrifugally stabilized, shallow spherical shells,[1,2] it was necessary to make a comprehensive study of the shell equilibrium stress and displacement distributions due to spin. Limitations of previous work on this problem have been surveyed in Refs. 1, 2, and 3. Some of the prior investigations[4,5] have used membrane theory which yields inconsistent results in the limiting case of the shallow spherical shell with infinite radius of curvature, the flat circular disk.

The shell configuration considered here is shown in Fig. 1. The thin, shallow spherical shell, spinning about its polar axis, has a peripheral radius b, and is fully clamped by a central hub of radius a. The outer edge of the shell is free. The defining equilibrium equations, based on Reissner's nonlinear shallow shell theory[6] may be written in terms of first central differences as a two-dimensional vector difference equation[2]

$$A_iT_{i-1} + B_iT_i + C_iT_{i+1} = D_i + E_i \qquad (1)$$

$$(i = 2,3, \ldots, n - 1)$$

where

$$T_j = \begin{bmatrix} g_j \\ f_j \end{bmatrix}$$

is the unknown column vector containing g_j and f_j, the dimensionless displacement variable and stress function, respectively, and where

$$A_i = \begin{bmatrix} (1/\Delta^2 - 1/2x\Delta)x\Delta^2 & 0 \\ 0 & (1/\Delta^2 - 1/2x\Delta)x\Delta^2 \end{bmatrix}$$
$$B_i = \begin{bmatrix} (-2/\Delta^2 - 1/x^2)x\Delta^2 & \lambda^4x\Delta^2 \\ -x\Delta^2 & (-2/\Delta^2 - 1/x^2)x\Delta^2 \end{bmatrix}$$
$$C_i = \begin{bmatrix} (1/\Delta^2 - 1/2x\Delta)x\Delta^2 & 0 \\ 0 & (1/\Delta^2 - 1/2x\Delta)x\Delta^2 \end{bmatrix} \qquad (2)$$
$$D_i = \begin{bmatrix} 0 \\ -x^2\Delta^2 \end{bmatrix}; \quad E_i = \begin{bmatrix} \gamma\lambda^4f_ig_i\Delta^2 \\ -\frac{1}{2}\gamma g_ig_i\Delta^2 \end{bmatrix}$$

The boundary conditions at the hub and at the free edge of the shell may be written in terms of first forward and first

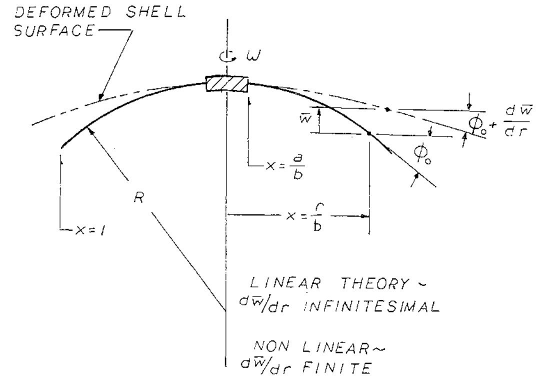

Fig. 1 Geometry and coordinate system for the spinning shallow spherical shell.

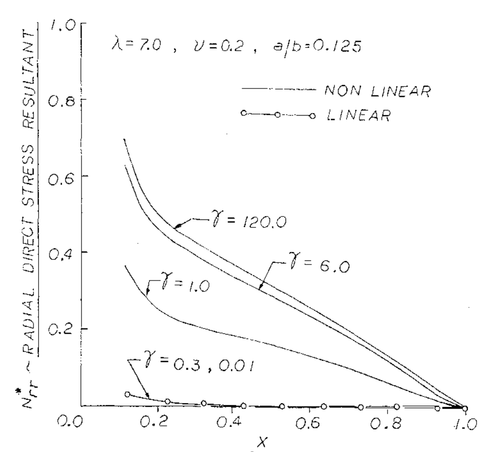

Fig. 2 Nondimensional radial direct stress resultant for a spinning shallow spherical shell, $N_{rr}^* = N_{rr}/\rho\omega b^2$. Shell geometry parameter = λ^4 = 2401.0; Poisson's ratio = ν = 0.20.

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